STRUCTURAL PROPERTIES OF COSTAS ARRAYS

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ABSTRACT. We apply combinatorial arguments to establish structural constraints on Costas arrays. We prove restrictions on when a Costas array can contain a large corner region whose entries are all 0. In particular, we prove a 2010 conjecture due to Russo, Erickson and Beard. We then constrain the vectors joining pairs of 1s in a Costas array by establishing a series of results on its number of "mirror pairs," namely pairs of these vectors having the same length but opposite slopes.

1. INTRODUCTION

This paper deals with structural properties of Costas arrays.

Costas arrays were introduced in 1965 by J. P. Costas as a means of improving the performance of radar and sonar systems [6]. A permutation array A of order n is a *Costas array* if the vectors formed by joining pairs of 1s in A are all distinct. We use the standard labelling convention for arrays (first index downwards, second index rightwards, both indices start from 1). It is sufficient to consider only one of the vectors joining each pair of 1s in a Costas array; by convention, we choose the vector pointing rightwards. We follow other authors in also using the conflicting labelling convention for vectors (first component horizontal, second component vertical), leading to Definition 1.1.

Definition 1.1. *The vector between '1' entries $A_{i,j}$ and $A_{k,\ell}$ of a permutation array $(A_{i,j})$, where $j < \ell$, is $(\ell - j, k - i)$.*

The central research questions for Costas arrays are: For which orders does a Costas array exist? How many inequivalent Costas arrays of each order are there? How can the Costas arrays of each order be constructed algebraically? A complete answer to each of these questions seems remote, despite nearly fifty years of study. Only two algebraic constructions (the Welch and Golomb constructions) and their variants are known, and the

vast majority of the exhaustively determined Costas arrays (up to order 29) are not explained by any known construction [19]. The smallest order for which the existence of a Costas array is open is 32, and this has been the case since 1984 [18]. However, settling the existence question for this order through exhaustive search remains well out of computational reach, with an estimated search time of 45,000 processor years in 2011 [10].

Researchers have instead sought to gain understanding by establishing structural constraints on Costas arrays, often inspired by examination of the known Costas arrays (see [15], [7], [9], for example). This approach appears to be difficult, especially for arbitrary Costas arrays rather than those that are algebraically constructed, and few such results have been found. Another approach is to impose an additional constraint such as diagonal symmetry or G-symmetry on a Costas array in order to facilitate theoretical analysis and exhaustive computational search [3], [20], [8], [23].

In this paper we apply combinatorial arguments to prove new structural constraints on Costas arrays.

Section 2 describes a key auxiliary 1985 result, due independently to Taylor [22] and to Freedman and Levanon [15], that two Costas arrays of order at least 4 always have a vector (joining pairs of 1s) in common. We give a complete proof of this result, which illustrates several ideas used elsewhere in the paper.

Section 3 studies Costas arrays that contain a large corner region whose entries are all 0. In the case of even order, when the corner region is an entire quadrant of the array, we use the Taylor-Freedman-Levanon result to prove a 2010 conjecture due to Russo, Erickson and Beard [21]. We then use techniques due to Erdős and Turán [12] and to Erdős *et al.* [11], following the outline provided by Blackburn *et al.* [2] and by Etzion [14], to demonstrate the asymptotic nonexistence of a Costas array of order n having an all-zero corner region whose side length grows linearly with n.

Section 4 introduces the concept of "mirror pairs" in permutation (and, in particular, Costas) arrays, namely pairs of vectors having the same length but opposite slopes. We again use the Taylor-Freedman-Levanon result to constrain the vectors contained in a single Costas array. We prove results for arbitrary permutation arrays and for G-symmetric Costas arrays, and outline results for algebraically constructed Costas arrays. These structural results complement those due to Drakakis, Gow and Rickard [9], which constrain the vectors contained in two different arbitrary Costas arrays. A further motivation for studying mirror pairs is that, if sufficiently strong results can be found, the computational burden for determining whether or not a Costas array of order 32 exists could be significantly reduced.

2. The Taylor-Freedman-Levanon result

In this section we prove the Taylor-Freedman-Levanon result on common vectors in Costas arrays, as Theorem 2.5. This result was proved by Taylor [22] by considering the Lee distance between pairs of 1s in a Costas array as well as the difference triangle of its corresponding permutation, and by Freedman and Levanon [15] by considering the cross-correlation function of the two Costas arrays. We instead use a proof due to Drakakis, Gow and Rickard [8] involving just the difference triangle. We associate a permutation array $(A_{i,j})$ with the permutation $\alpha \in \mathcal{S}_n$ by the convention that $A_{i,j} = 1$ if and only if $\alpha(j) = i$.

Definition 2.1. *The* difference triangle $T(\alpha)$ *of* $\alpha \in S_n$ *is* $(t_{i,j}(\alpha))$, *where*

$$t_{i,j}(\alpha) = \alpha(i+j) - \alpha(j) \quad \textit{for } 1 \le i < n,\ 1 \le j \le n - i.$$

We use the standard labelling convention for arrays (first index downwards, second index rightwards) to number the rows and columns of the difference triangle. For example, Figure 1 shows the difference triangle of the permutation $\alpha = [3, 1, 6, 2, 5, 4]$, in which row 2 is the sequence $(3, 1, -1, 2)$ and column 4 is the sequence $(3, 2)$. The antidiagonals are the sequences $(-2), (3, 5), \ldots, (1, 3, -2, 2, -1)$.

−2	5	−4	3	−1
3	1	−1	2	
−1	4	−2		
2	3			
1				

FIGURE 1. Difference triangle of the permutation $[3, 1, 6, 2, 5, 4]$

In preparation for the proof of Theorem 2.5, we give some basic properties of the difference triangle of a permutation. The combination of Lemma 2.2(i) (which is elementary) with Lemma 2.2(ii), due to Costas [4], shows that the difference triangle is a setting in which the defining property of a Costas array appears mathematically natural.

Lemma 2.2. *Let* α *be a permutation.*

(i) *No column of $T(\alpha)$ and no antidiagonal of $T(\alpha)$ contains a repeated value.*

(ii) *No row of $T(\alpha)$ contains a repeated value if and only if the permutation array corresponding to α is a Costas array.*

Proof. Part (i) holds because α is a permutation. Part (ii) holds because the permutation array corresponding to α contains the vector (w, h) starting from position $(\alpha(j), j)$ if and only if $t_{w,j}(\alpha) = h$. □

Lemma 2.3 ([5]). *The difference triangle of $\alpha \in S_n$ contains exactly $n - k$ entries from $\{-k, k\}$, for each k satisfying $1 \le k < n$.*

Proof. From the n entries of α we can form exactly $n - k$ pairs whose values differ in magnitude by k (namely the pairs $\{\ell, k + \ell\}$ for $1 \le \ell \le n - k$). $\qquad\square$

The following lemma specifies relations between certain elements of the difference triangle. It follows directly from the definition of the difference triangle and, for part (i), from the permutation property of α. A more general version of the lemma is given in [24, Section 4.2].

Lemma 2.4. *Let $\alpha \in S_n$. Then*

(i) *for $n \ge 3$ and $1 \le c \le n - 2$,*

$$t_{n-1,1}(\alpha) = t_{n-1-c,1}(\alpha) + t_{n-1-c,1+c}(\alpha) \quad \textit{if and only if} \quad n = 2c + 1$$

(ii) *for $n \ge 4$,*

$$t_{n-3,2}(\alpha) = t_{n-2,1}(\alpha) + t_{n-2,2}(\alpha) - t_{n-1,1}(\alpha).$$

Theorem 2.5 (Taylor [22], Freedman-Levanon [15]). *Every pair of Costas arrays of order $n \ge 4$ has a vector in common.*

Proof. [8] Suppose, for a contradiction, that α, β are permutations corresponding to Costas arrays of order $n \ge 4$ having no vector in common. For $1 \le w < n$, let $t_w(\alpha)$ and $t_w(\beta)$ be the set of elements contained in row w of $T(\alpha)$ and $T(\beta)$, respectively, and let $t_w(\alpha, \beta)$ be the multiset union of $t_w(\alpha)$ and $t_w(\beta)$. By assumption and by Lemma 2.2(ii), for each w the multiset $t_w(\alpha, \beta)$ has no repeated elements and so is actually a set.

For $1 \le k < n$, we now prove by induction on k that

$$(2.1) \qquad\qquad -k, k \in t_w(\alpha, \beta) \quad \text{for } w = 1, 2, \ldots, n - k.$$

For the base case $k = 1$, we know from Lemma 2.3 that there are a total of $2(n - 1)$ entries from $\{-1, 1\}$ distributed over the $n - 1$ sets $\{t_w(\alpha, \beta) : 1 \le w \le n - 1\}$. Since each of these sets contains no repeated elements, we conclude that $-1, 1 \in t_w(\alpha, \beta)$ for $w = 1, 2, \ldots, n - 1$. This establishes the base case. Assume now that the cases up to $k - 1$ hold. By Lemma 2.3, there are a total of $2(n - k)$ entries from $\{-k, k\}$ distributed over the $n - 1$ sets $\{t_w(\alpha, \beta) : 1 \le w \le n - 1\}$. But the inductive hypothesis implies that the sets $\{t_w(\alpha, \beta) : n - k \le w \le n - 1\}$ contain no elements from $\{-k, k\}$, so the $2(n - k)$ entries from $\{-k, k\}$ are in fact distributed over the $n - k$ sets $\{t_w(\alpha, \beta) : 1 \le w \le n - k\}$. Since each of these sets contain no repeated elements, we conclude that $-k, k \in t_w(\alpha, \beta)$ for $w = 1, 2, \ldots, n - k$. This completes the induction.

It follows from (2.1) that $t_{n-k}(\alpha, \beta) = \{-k, \ldots, -1, 1, \ldots k\}$ for $1 \le k < n$ and, in particular, that

$$(2.2) \qquad\qquad t_{n-1}(\alpha, \beta) = \{-1, 1\},$$

$$(2.3) \qquad\qquad t_{n-2}(\alpha, \beta) = \{-2, -1, 1, 2\},$$

$$(2.4) \qquad\qquad t_{n-3}(\alpha, \beta) = \{-3, -2, -1, 1, 2, 3\}.$$

We may assume from (2.2) that $t_{n-1}(\alpha) = \{1\}$ and $t_{n-1}(\beta) = \{-1\}$. Lemma 2.2(i) then implies that $1 \notin t_{n-2}(\alpha)$ and $-1 \notin t_{n-2}(\beta)$. Now $t_{n-2}(\alpha) \neq \{-2, 2\}$ since otherwise (2.3) would imply the contradiction $-1 \in t_{n-2}(\beta)$, and $t_{n-2}(\alpha) \neq \{-1, 2\}$ otherwise Lemma 2.4(i) with $c = 1$ would imply the contradiction $n = 3$. Therefore $t_{n-2}(\alpha) = \{-2, -1\}$, by (2.3). Finally, Lemma 2.4(ii) shows that $t_{n-3,2}(\alpha) = -2 - 1 - 1 = -4$, contradicting (2.4). $\qquad \square$

3. All-zero corner regions

Definition 3.1. *An* all-zero corner region *of a permutation array A is a square subarray, whose entries are all 0, containing one of the four corner elements of A.*

The size of an all-zero corner region of a permutation array of order n is at most $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$, by the permutation property. In this section, we show that Costas arrays of even order n attain this upper bound only for small orders (Theorem 3.3). We then use techniques due to Erdős and Turán [12] and to Erdős *et al.* [11] to show that, for fixed $c > 0$, asymptotically no Costas array of order n with an all-zero corner region of size $\lfloor cn \rfloor \times \lfloor cn \rfloor$ exists (Theorem 3.5).

We begin with even n and a corner region of size $\frac{n}{2} \times \frac{n}{2}$, which is simply a quadrant. Figure 2 shows examples of Costas arrays of order 2, 4 and 6 containing two diagonally opposite all-zero quadrants. It was conjectured in 2010 that there are no larger such examples.

Conjecture 3.2 (Russo, Erickson and Beard [21])**.** *No Costas array of even order greater than* 6 *contains two all-zero quadrants.*
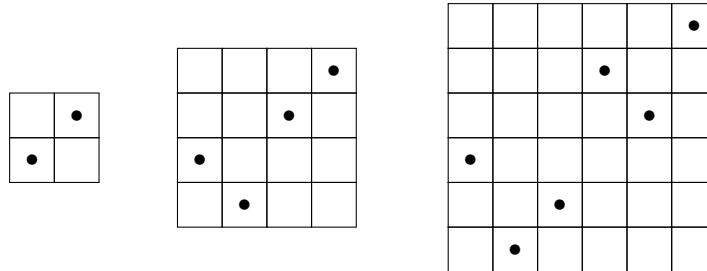

FIGURE 2. Costas arrays with two all-zero quadrants

We now use Theorem 2.5 to prove a result stronger than the statement of Conjecture 3.2.

Theorem 3.3. *No Costas array of even order greater than* 6 *contains an all-zero quadrant.*

Proof. Suppose, for a contradiction, that A is a Costas array of order $2m > 6$ containing an all-zero quadrant. Since two diagonally opposite quadrants of a permutation array of even order must contain equally many 1s, the array A

has two all-zero quadrants in diagonally opposite positions. Each of the other two quadrants then forms a Costas array of order $m > 3$. By Theorem 2.5, these two Costas arrays contain a common vector. This contradicts the Costas property for A. □

Corollary 3.4 (Conjecture 3.2 holds). *No Costas array of even order greater than 6 contains two all-zero quadrants.*

We now demonstrate the asymptotic nonexistence of a Costas array of order n having an all-zero corner region whose side length grows linearly with n. Theorem 3.5 below occurs as a special case of a result stated without proof by Blackburn *et al.* [2, Section IV.C] and by Etzion [14, Theorem 3]. Although a careful reading of the text of [2, Section IV] can be used to re-construct a proof of that result, the special case required here (Theorem 3.5) can be proved directly by means of a short and simple argument that is self-contained (apart from reference to an elementary result [2, Lemma 13]). As mentioned in [2] and [14], the underlying method is due to Erdős and Turán [12] and to Erdős *et al.* [11].

Theorem 3.5. *Let $c > 0$ be a fixed real number. For all sufficiently large n, there is no Costas array of order n containing an all-zero corner region of size $\lfloor cn \rfloor \times \lfloor cn \rfloor$.*

Proof. Suppose, for a contradiction, that A is a Costas array of order n containing an all-zero corner region of size $\lfloor cn \rfloor \times \lfloor cn \rfloor$ for arbitrarily large n. Place A on an infinite grid of unit squares in the plane, with the centre of each cell of A lying on a grid point, and regard each '0' and '1' entry of A as being written at the centre of its associated cell. Let A' be the region of the plane comprising the cells of A in which the '1' entries of A are constrained to lie, namely an $n \times n$ square that is missing a $\lfloor cn \rfloor \times \lfloor cn \rfloor$ corner square.

Consider the w circles of radius $n^{\frac{2}{3}}$ whose centre lies at the centre of a grid square and which intersect the region A'. The centre of any such circle lies within distance $n^{\frac{2}{3}}$ of some point of A', and so the grid squares associated with the centres of these w circles are all completely contained within an $(n + 2n^{\frac{2}{3}} + 1) \times (n + 2n^{\frac{2}{3}} + 1)$ region that is missing a $\lfloor cn \rfloor \times \lfloor cn \rfloor$ corner region. Since each grid square has unit area, we therefore have

$$(3.1) \qquad w \le (n + 2n^{\frac{2}{3}} + 1)^2 - \lfloor cn \rfloor^2 .$$

For $1 \le i \le w$, let m_i be the number of '1' entries of A contained in the ith circle and write μ for the mean of the m_i. For $1 \le j \le n$, let a_j be the number of circles containing the jth '1' entry of A, so that

$$(3.2) \qquad w\mu = \sum_{j=1}^{n} a_j .$$

We next show that the sum $S := \sum_{i=1}^{w} \binom{m_i}{2}$ satisfies the inequality

$$(3.3) \qquad w\binom{\mu}{2} \le S \le \binom{a}{2},$$

where a is the number of grid points contained in a single circle. The left inequality of (3.3) is given by rearrangement of the inequality $\sum_{i=1}^{w}(m_i-\mu)^2 \geq 0$ and the definition of μ. For the right inequality of (3.3), note that S counts the number of vectors formed by joining pairs of 1s contained in a circle, summed over all w circles. If we superimpose all w circles then, by the Costas property, each such vector occurs at most once. Therefore S is at most the number of vectors joining two grid points contained in a single circle, namely $\binom{a}{2}$. This establishes (3.3).

Discard S from (3.3) and substitute for μ from (3.2). Rearrange to make w the subject, and then combine with (3.1) to give

$$(3.4) \qquad \frac{(\sum_{j=1}^{n} a_j)^2}{a(a-1) + \sum_{j=1}^{n} a_j} \leq (n + 2n^{\frac{2}{3}} + 1)^2 - \lfloor cn \rfloor^2.$$

Since the number of grid points contained in a circle of radius ℓ is $\pi\ell^2 + O(\ell)$ [2, Lemma 13], we have $a = \pi n^{\frac{4}{3}} + O(n^{\frac{2}{3}})$ and similarly $a_j = \pi n^{\frac{4}{3}} + O(n^{\frac{2}{3}})$. Therefore, to leading order in n, the left side of (3.4) is n^2. On the other hand, to leading order in n, the right side of (3.4) is $n^2(1 - c^2)$. This gives the required contradiction as $n \to \infty$. □

4. Mirror pairs

In this section we introduce a new structural constraint on permutation (and, in particular, Costas) arrays.

Transformation of a Costas array under the action of the dihedral group D_4 gives an equivalent Costas array. By applying Theorem 2.5 to a Costas array A of order at least 4, and its image under reflection in a vertical axis, we can conclude that A must contain a pair of related vectors. (We can guarantee additional pairs of related vectors by applying Theorem 2.5 to A and its image under $90°$ or $270°$ rotation, but not necessarily to A and its image under diagonal or antidiagonal reflection.) This motivates the following definition and proposition.

Definition 4.1. *A permutation array contains a* mirror pair *of width $w > 0$ and height $h > 0$ (abbreviated as a (w,h)-mirror pair) if it contains vectors (w, h) and $(w, -h)$.*

Proposition 4.2. *Every Costas array of order $n \geq 4$ contains a mirror pair.*

Proof. Let A be a Costas array of order $n \geq 4$. By Theorem 2.5, A and its image under reflection in a vertical axis have a common vector, say (w, h). Then A contains the vector $(w, -h)$. □

For example, the Costas array of order 7 shown in Figure 3 contains a $(3,1)$-mirror pair. The condition $n \geq 4$ in Proposition 4.2 is necessary: the endpoints of the vectors (w, h) and $(w, -h)$ must involve distinct '1' entries, otherwise the permutation property of the Costas array would not hold.

The existence of mirror pairs in Costas arrays is a structural property that does not appear to have been observed previously. In this section, we
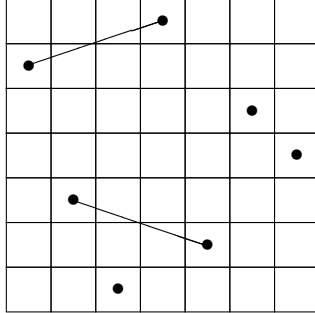
FIGURE 3. A $(3,1)$-mirror pair in a Costas array of order 7

constrain the vectors of a Costas array by analysing the number of its mirror pairs and their parameters w and h.

We begin with an upper bound on the number of mirror pairs of width w in a permutation array.

Proposition 4.3. *Let A be a permutation array of order $n \geq 4$ containing m_w mirror pairs of width w. Then*

(i) $m_1 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

(ii) $m_w \leq \left\lfloor \frac{n-w}{2} \right\rfloor$ *for each w satisfying $2 \leq w \leq n-1$.*

(iii) *The total number of mirror pairs in A is at most $\frac{n(n-2)}{4}$ for n even, and at most $\frac{(n+1)(n-3)}{4}$ for n odd.*

Proof. For parts (i) and (ii), let $1 \leq w \leq n-1$. Since A is a permutation array, it contains exactly $n - w$ vectors with width (first component) w and therefore at most $\left\lfloor \frac{n-w}{2} \right\rfloor$ mirror pairs of width w. It is now sufficient to show that A cannot contain $\frac{n-1}{2}$ mirror pairs of width 1 when n is odd. Suppose otherwise, so that the $n - 1$ vectors of A with width 1 can be arranged into mirror pairs of height $h_1, h_2, \ldots, h_{\frac{n-1}{2}}$, and let the '1' entry in column 1 of A occur in row i. Then the '1' entry in column n of A occurs in row $i + \sum_k h_k + \sum_k (-h_k) = i$, contradicting the permutation property.

Part (iii) is given by summing the bounds of parts (i) and (ii) over w. □

Numerical data obtained from analysis of the database [19] of Costas arrays, presented in Figure 4, suggest that the actual number of mirror pairs in a Costas array of order n broadly increases with n, and that its mean grows faster than linearly with n. In Section 4.1 we shall strengthen Proposition 4.3 for G-symmetric Costas arrays by using their additional structure to establish lower and upper bounds on the number of mirror pairs of various widths. In Section 4.2 we shall fix precisely the number of
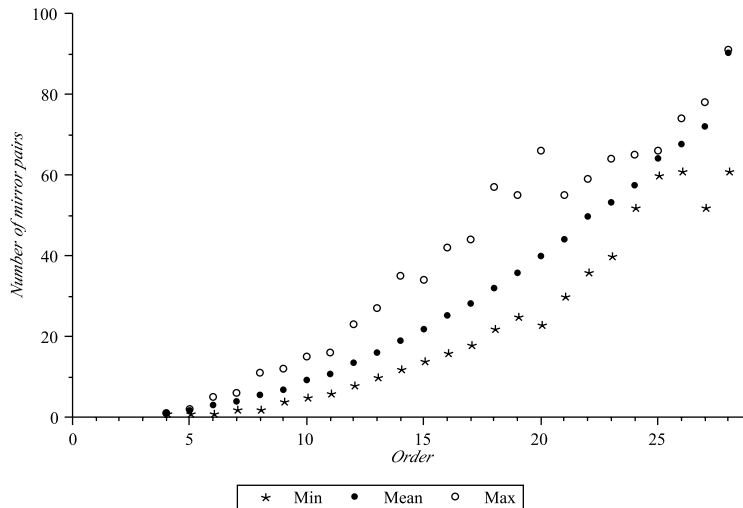
FIGURE 4. Number of mirror pairs in Costas arrays up to order 29

mirror pairs of every width in Welch Costas arrays, and state other existence results for mirror pairs in Welch Costas and Golomb Costas arrays.

The large number of vectors having small width suggests that small values of w might be more likely to admit mirror pairs of width w. This leads us to pay particular attention to mirror pairs of small width and, by similar reasoning, those of small height. Analysis of the database [19] of Costas arrays shows that there is at least one width 1 mirror pair and at least one height 1 mirror pair in each Costas array with $4 \leq n \leq 8$, with the exception of the Costas arrays corresponding to the permutations $[3, 1, 2, 4]$, $[2, 3, 5, 1, 4]$, $[2, 6, 4, 5, 1, 3]$, $[5, 3, 2, 6, 1, 4]$, $[2, 5, 1, 6, 4, 3]$ $[3, 1, 6, 3, 5, 4]$, $[2, 6, 3, 8, 1, 7, 5, 4]$ and $[2, 8, 1, 6, 5, 3, 7, 4]$ and their equivalence classes. (Of these, only that corresponding to $[5, 3, 2, 6, 1, 4]$ lacks both a width 1 mirror pair and a height 1 mirror pair.) Moreover, there is at least one width 1 mirror pair and at least one height 1 mirror pair in every Costas array with $9 \leq n \leq 29$. These observations prompt the following question.

Question 4.4. *Does every Costas array of order $n \geq 9$ contain a mirror pair of width 1 and a mirror pair of height 1?*

We can simplify Question 4.4 by noting the action of D_4 on the mirror pairs of a permutation array.

Remark 4.5. *Suppose that a permutation array contains a (w, h)-mirror pair. Then so does its image under horizontal reflection, vertical reflection and rotation by $180°$. Its image under diagonal reflection, antidiagonal reflection and rotation by $90°$ and $270°$ contains an (h, w)-mirror pair.*

We see that in order to answer Question 4.4 with *yes*, it would be sufficient to show that there is at least one width 1 mirror pair in each Costas array of order $n \geq 9$ (not just in the equivalence class of each Costas array). Indeed, by Remark 4.5, there would then also be at least one height 1 mirror pair in each such Costas array. Numerical data presented in Figure 5 suggest that the number of width 1 mirror pairs in a Costas array of order n grows with n, providing evidence that the answer to Question 4.4 is *yes*. Figure 5 also shows that the upper bound on the number m_1 of width 1 mirror pairs, given in Proposition 4.3(i), is attained for all n in the range $4 \leq n \leq 29$ except 24, 25, and 26. We shall see in Theorem 4.10(i) that all G-symmetric Costas arrays of order n attain this upper bound.
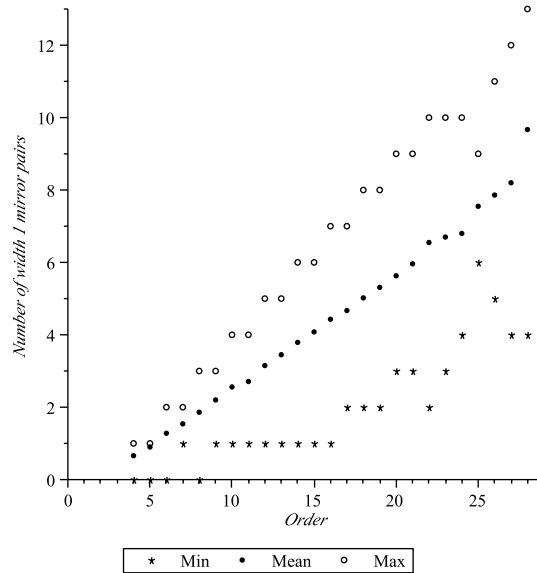

FIGURE 5. Number m_1 of width 1 mirror pairs in Costas arrays up to order 29

Analysis of the Costas array database [19] also shows that for $14 \leq n \leq 29$ every Costas array of order n has a mirror pair of width 2 and therefore, by Remark 4.5, a mirror pair of height 2. This prompts the following question.

Question 4.6. *Does every Costas array of order $n \geq 14$ have a mirror pair of width 2 and a mirror pair of height 2?*

Questions 4.4 and 4.6 are more easily studied for Costas arrays that are algebraically constrained (G-symmetric Costas arrays) or constructed (Welch Costas arrays and Golomb Costas arrays). We will examine the mirror pairs of small width and height in these classes of Costas arrays in Sections 4.1 and 4.2. Before doing so, we will provide a partial answer to Questions 4.4

and 4.6 for all Costas arrays, without imposing any such algebraic restrictions, in Theorem 4.7.

Some of our proofs make extensive use of the difference triangle. By Definition 2.1, a Costas array with corresponding permutation α contains a (w, h)-mirror pair if and only if both $-h$ and h appear in row w of $T(\alpha)$.

Theorem 4.7. *Every Costas array of order $n \geq 6$ contains a mirror pair of width 1 or 2 and a mirror pair of height 1 or 2.*

Proof. By Remark 4.5, it is sufficient to show that every Costas array of order $n \geq 6$ contains a mirror pair of height 1 or 2. Suppose, for a contradiction, that A is a Costas array of order $n \geq 6$, with corresponding permutation α, containing neither a mirror pair of height 1 nor a mirror pair of height 2. Then no row of $T(\alpha)$ contains more than one entry from $\{-1, 1\}$ and no row of $T(\alpha)$ contains more than one entry from $\{-2, 2\}$. Since $T(\alpha)$ contains exactly $n-1$ entries from $\{-1, 1\}$ by Lemma 2.3, it therefore contains exactly one entry from $\{-1, 1\}$ in each of its $n-1$ rows. This accounts for the single entry of row $n-1$ of $T(\alpha)$; and since $T(\alpha)$ contains exactly $n-2$ entries from $\{-2, 2\}$ by Lemma 2.3, it must then contain exactly one entry from $\{-2, 2\}$ in each of its first $n-2$ rows.

Write row i of $T(\alpha)$ as $r_i(\alpha)$. By Remark 4.5, both rotation of A through $180°$ and reflection of A in a horizontal axis leave the mirror pairs of A unchanged, but the first transformation reflects the rows of $T(\alpha)$ and the second negates the entries of $T(\alpha)$. We may therefore assume that $r_{n-1}(\alpha) = (1)$ and $r_{n-2}(\alpha) = (x, y)$, where $x \in \{-1, 1\}$ and $y \in \{-2, 2\}$. Lemma 2.2(i) then gives $x = -1$, and Lemma 2.4(i) with $c = 1$ gives $y = -2$. Then by Lemma 2.4(ii), $r_{n-3}(\alpha) = (u, -4, v)$, where $\{|u|, |v|\} = \{1, 2\}$. By Lemma 2.2(i) we have $u \notin \{-1, 1\}$ and $v \neq 1$, so $(u, v) = (2, -1)$ or $(-2, -1)$. But by Lemma 2.4(i) with $c = 2$, we have $u + v \neq 1$ since $n \geq 6$. This forces $(u, v) = (-2, -1)$, so the last three rows of $T(\alpha)$ are as shown below.

-2	-4	-1
-1	-2	
1		

From these entries of $T(\alpha)$ and the assumption $n \geq 6$ we obtain $\alpha = [m, m+3, m+2, \ldots, m-2, m-1, m+1]$ for some m. Then the first row of $T(\alpha)$ contains both $(m+2) - (m+3) = -1$ and $(m-1) - (m-2) = 1$, which is a contradiction. □

By examining all Costas arrays of order 4 and 5, we can extend Theorem 4.7 to $n \geq 4$, with the exception of the Costas array corresponding to the permutation $[2, 3, 5, 1, 4]$ and its equivalence class.

4.1. **Mirror pairs in G-symmetric Costas arrays.**

Definition 4.8. *An $n \times n$ array corresponding to the permutation γ is G-symmetric if*

$$n \text{ is even and } \gamma\left(j + \frac{n}{2}\right) + \gamma(j) = n + 1 \text{ for } 1 \le j \le \frac{n}{2}$$

or

$$n \text{ is odd and } \gamma\left(\frac{n+1}{2}\right) = \frac{n+1}{2} \text{ and } \gamma\left(j + \frac{n+1}{2}\right) + \gamma(j) = n+1 \text{ for } 1 \le j \le \frac{n-1}{2}.$$

In this section, we consider mirror pairs in G-symmetric Costas arrays of order n. We show the answer to Question 4.4 for this class of Costas arrays is *yes* for even n (Theorem 4.10(i) and Theorem 4.13). We also provide a partial answer to Question 4.4 for odd n by showing the existence of a width 1, but not necessarily a height 1, mirror pair for arrays in this class (Theorem 4.10(i)). We likewise provide a partial answer to Question 4.6 for all n for arrays in this class (Theorem 4.10(ii)). Furthermore, we constrain the total number of mirror pairs in a G-symmetric Costas array more strongly than in Proposition 4.3(iii) (Theorem 4.10(v)) by proving new lower and upper bounds for various widths.

We call '1' entries of a G-symmetric array that are separated by $\lfloor \frac{n+1}{2} \rfloor$ columns *G-symmetric images* of each other. We require the following lemma in the proof of Theorem 4.10.

Lemma 4.9. *Let G be a G-symmetric Costas array of order $n \ge 4$, and let L and R denote the leftmost $\lfloor \frac{n}{2} \rfloor$ and rightmost $\lfloor \frac{n}{2} \rfloor$ columns of G, respectively.*

  (i) *Every vector joining '1' entries in L forms a mirror pair with the vector in R that joins the G-symmetric images of its endpoints.*
 (ii) *If one vector of a mirror pair in G joins a '1' entry in L to a '1' entry in R then so does the other. The G-symmetric images of the endpoints of two such vectors that form a (w, h)-mirror pair are the endpoints of two such vectors that form a $(2\lfloor \frac{n+1}{2} \rfloor - w, h)$-mirror pair.*

Proof. For part (i), let $1 \le w \le \lfloor \frac{n}{2} \rfloor - 1$ and $1 \le j \le \lfloor \frac{n}{2} \rfloor - w$. By G-symmetry, the vector in L joining the '1' entries in columns j and $j + w$ forms a mirror pair of width w with the vector in R joining the '1' entries in columns $j + \lfloor \frac{n+1}{2} \rfloor$ and $j + w + \lfloor \frac{n+1}{2} \rfloor$. Part (ii) follows from part (i), the Costas property and G-symmetry. $\square$

Theorem 4.10. *Let G be a G-symmetric Costas array of order $n \ge 4$ containing m_w mirror pairs of width w. Then*

  (i) $m_1 = \lfloor \frac{n}{2} \rfloor - 1$.

 (ii) $m_w \ge \lfloor \frac{n}{2} \rfloor - w$ *for each w satisfying $2 \le w \le \lfloor \frac{n}{2} \rfloor - 1$.*

(iii) $m_w \le \lfloor \frac{n-w}{2} \rfloor$ *for each w satisfying $2 \le w \le n - 1$.*

 (iv) $m_{\lfloor \frac{n+1}{2} \rfloor} = 0$.

(v) *For even n, the total number of mirror pairs in G lies in the interval*

$$\left[\frac{n(n-2)}{8}, \ \frac{n(n-2)}{4} - \left\lfloor \frac{n}{4} \right\rfloor \right]$$

and has the same parity as $\frac{n(n-2)}{8}$. For odd n, the total number of mirror pairs in G lies in the interval

$$\left[\frac{(n-1)(n-3)}{8}, \ \frac{(n+1)(n-3)}{4} - \left\lfloor \frac{n-1}{4} \right\rfloor \right].$$

Proof. For parts (i) and (ii), let $1 \leq w \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Lemma 4.9(i) accounts for $\left\lfloor \frac{n}{2} \right\rfloor - w$ mirror pairs of width w to give part (ii), and combination with Proposition 4.3(i) gives part (i).

Part (iii) restates Proposition 4.3(ii).

For part (iv), since the endpoints of a vector of width $\left\lfloor \frac{n+1}{2} \right\rfloor$ in G are G-symmetric images of each other, their vertical separation varies with the smaller of the row numbers in which they occur. Therefore no two distinct vectors of width $\left\lfloor \frac{n+1}{2} \right\rfloor$ in G have endpoints with the same vertical separation, and so G does not contain a mirror pair of width $\left\lfloor \frac{n+1}{2} \right\rfloor$.

For part (v), the intervals for the total number of mirror pairs in G are given by combining all previous parts. For even n, the lower limit of the interval is a count via Lemma 4.9(i) of the mirror pairs involving a vector joining '1' entries in the left half of G and a vector joining '1' entries in the right half of G. Since this count exhausts every such vector, all other mirror pairs must involve vectors joining a '1' entry in the left half of G to a '1' entry in the right half of G. Then by Lemma 4.9(ii), these additional mirror pairs can be associated using G-symmetry into pairs, and by part (iv) no mirror pair is associated with itself. This gives the parity condition. $\square$

The upper bound given in Theorem 4.10(v) on the total number of mirror pairs in a G-symmetric Costas array of order n is strictly larger than the lower bound for all $n > 4$. This upper bound is attained by a G-symmetric Costas array of order n in the case $n = 5$ (corresponding permutation $[2, 5, 3, 4, 1]$, total two mirror pairs), the case $n = 6$ (corresponding permutation $[1, 5, 3, 6, 2, 4]$, total five mirror pairs), the case $n = 8$ (corresponding permutations $[1, 8, 6, 3, 7, 2, 4, 5]$, $[1, 6, 7, 4, 8, 3, 2, 5]$, $[7, 1, 2, 8, 4, 6, 5, 3]$, total ten mirror pairs each), but in no other case in the range $8 < n \leq 28$ (by reference to the Costas array database [1] for $n < 28$, and [10], [23] for $n = 28$).

We noted previously that if every Costas array of order n contains a width 1 mirror pair then, by Remark 4.5, it also contains a height 1 mirror pair. However, we cannot conclude from Theorem 4.10(i) that every G-symmetric Costas array of order $n \geq 4$ contains even a single mirror pair of height 1, because G-symmetry is not preserved under the transpose operation. Nonetheless, we shall prove in Theorem 4.13 that (apart from some

small exceptions) every G-symmetric Costas array of even order indeed contains a height 1 mirror pair. We firstly establish two lemmas, in preparation for a proof by contradiction.

Lemma 4.11. *Suppose that G is a G-symmetric Costas array of even order n, corresponding to the permutation γ and containing no mirror pair of height 1. Then the first $\frac{n}{2}$ elements of γ all have the same parity.*

Proof. No vector of G of height 1 is completely contained in the left half or the right half of G, otherwise by Lemma 4.9(i) G would contain a mirror pair of height 1. Therefore no two '1' entries in the left half of G occur in consecutive rows, and no two '1' entries in the right half of G occur in consecutive rows. It follows that the '1' entry in row i of G occurs in the opposite half from the '1' entry in row $i + 1$. $\qquad\square$

We next constrain the difference triangle of a permutation γ satisfying the parity constraint in Lemma 4.11.

Lemma 4.12. *Let G be a G-symmetric Costas array of even order n, corresponding to the permutation γ, and suppose that the first $\frac{n}{2}$ entries of γ all have the same parity. Let T_1 and T_2 be the triangular regions of $T(\gamma)$ indicated below, each involving $\frac{n}{2} - 1$ rows.*
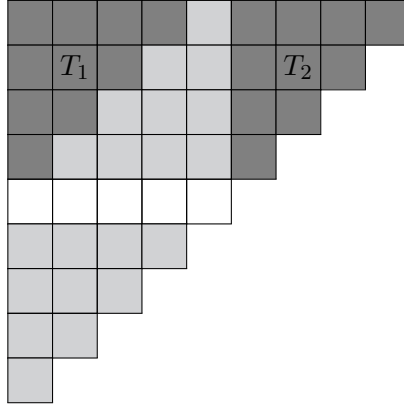
Then

(i) *all entries in T_1 are even, and $T_2 = -T_1$*

(ii) *for $1 \le w \le \frac{n}{2} - 1$, row w of T_1 contains exactly one element from each of $\{-2, 2\}, \{-4, 4\}, \ldots, \{-(n - 2w), n - 2w\}$.*

Proof. For (i), let $1 \le w \le \frac{n}{2} - 1$ and $1 \le j \le \frac{n}{2} - w$. The (w, j) entry of T_1 is $\gamma(w + j) - \gamma(j)$, which is even because $\gamma(w + j)$ and $\gamma(j)$ have the same parity. The (w, j) entry of T_2 is $\gamma(w + j + \frac{n}{2}) - \gamma(j + \frac{n}{2}) = (n + 1 - \gamma(w + j)) - (n + 1 - \gamma(j))$ by G-symmetry, and so $T_2 = -T_1$.

For (ii), let $1 \le w \le \frac{n}{2} - 1$. For each k satisfying $1 \le k \le \frac{n}{2} - 1$, by Lemma 2.3 the difference triangle $T(\gamma)$ contains exactly $n - 2k$ elements from $\{-2k, 2k\}$. There are a total of $\frac{n}{2}(\frac{n}{2} - 1)$ of these even elements in $T(\gamma)$, and by (i) they are all contained in $T_1 \cup T_2$. We deduce from $T_2 = -T_1$ that, for each k

satisfying $1 \le k \le \frac{n}{2} - 1$, the triangle T_1 contains exactly $\frac{n}{2} - k$ elements from $\{-2k, 2k\}$ distributed over its $\frac{n}{2} - 1$ rows. By Lemma 2.2(ii) and $T_2 = -T_1$, no two such elements occur in the same row of T_1. The result now follows by a simple induction. $\qquad\square$

We can now classify the G-symmetric Costas arrays of even order $n \ge 4$ that do not contain a mirror pair of height 1.

Theorem 4.13. *The only G-symmetric Costas arrays of even order $n \ge 4$ that do not contain a mirror pair of height 1 are those corresponding to the permutations $[3, 1, 2, 4]$ and $[2, 6, 4, 5, 1, 3]$ and their images under horizontal reflection, vertical reflection and $180°$ rotation.*

Proof. Suppose that G is a G-symmetric Costas array of even order $n \ge 4$, corresponding to the permutation γ and containing no mirror pair of height 1. By Lemma 4.11, the conclusions of Lemma 4.12 hold.

By Lemma 2.3, $T(\gamma)$ has exactly $n - 1$ entries from $\{-1, 1\}$, and by assumption no two are in the same row. Therefore

(4.1)   each row of $T(\gamma)$ contains exactly one entry from $\{-1, 1\}$.

Since horizontal reflection, vertical reflection and $180°$ rotation preserve G-symmetry, we may assume that $t_{n-1,1}(\gamma) = 1$ and, using Lemma 2.2(i), that $t_{n-2,1}(\gamma) = -1$. Write $\gamma(1) = m$, so that $\gamma(n - 1) = m - 1$ and $\gamma(n) = m + 1$. Then $t_{1,n-1}(\gamma) = 2$, which by Lemma 4.12(i) gives $t_{1,\frac{n}{2}-1}(\gamma) = -2$.

For $n = 4$, this last conclusion reduces to $t_{1,1}(\gamma) = -2$ and so $\gamma = [m, m - 2, m - 1, m + 1]$, which forces $\gamma = [3, 1, 2, 4]$. For $n > 4$, Lemmas 4.12(ii) and 2.2(i) together give $t_{\frac{n}{2}-1,1}(\gamma) = 2$, and then Lemma 4.12(i) gives $t_{\frac{n}{2}-1,\frac{n}{2}+1}(\gamma) = -2$. For $n = 6$, this implies that $\gamma = [m, m + 4, m + 2, m + 3, m - 1, m + 1]$, which forces $\gamma = [2, 6, 4, 5, 1, 3]$. It is easily verified that in both these cases $n = 4$ and $n = 6$, G is a G-symmetric Costas array but its transpose is not, giving the eight exceptional Costas arrays.

Otherwise, for $n \ge 8$, we seek a contradiction. Write $x = t_{1,1}(\gamma)$ and $y = t_{n-3,2}(\gamma)$ (see Figure 6). By Lemma 2.2(ii) we have $x \ne -2$, so $\gamma(2) \ne m - 2$. Then $y = \gamma(n - 1) - \gamma(2) \ne 1$, and therefore $t_{n-3,3}(\gamma) = -1$ by (4.1) and repeated use of Lemma 2.2(i). This implies that $\gamma(3) = m + 2$, and so $t_{2,1}(\gamma) = 2$. This contradicts Lemma 2.2(i) because $2 \ne \frac{n}{2} - 1$. $\qquad\square$

4.2. **Mirror pairs in algebraically constructed Costas arrays.** In this section, we consider mirror pairs in the two main classes of algebraically constructed Costas arrays: Welch Costas and Golomb Costas arrays.

Theorem 4.14 (Welch Construction $W_1(p, \phi, c)$ [16])**.** *Let ϕ be a primitive element of $\mathbb{F}_p$, where p is a prime, and let c be a constant. Then the permutation array $(A_{i,j})$ of order $p - 1$ with*

$$A_{i,j} = 1 \quad \text{if and only if} \quad \phi^{j+c-1} \equiv i \pmod{p}$$
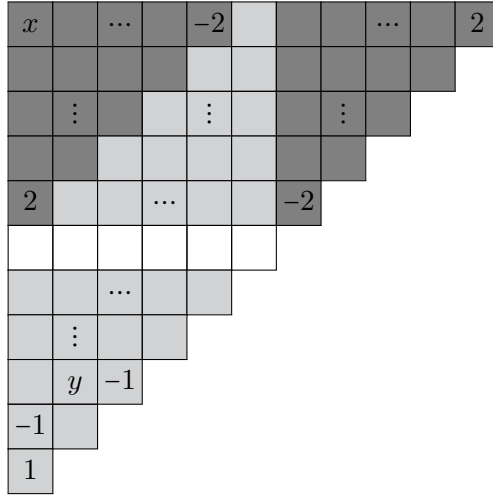
is a Costas array.

FIGURE 6. Difference triangle $T(\gamma)$ for the proof of Theorem 4.13

Varying the parameter c in the construction of Theorem 4.14 in the range $0, \ldots, p-2$ corresponds to cyclically shifting the columns of the Welch Costas array $(A_{i,j})$.

Theorem 4.15 (Golomb construction $G_2(q, \phi, \rho)$ [17])**.** *Let ϕ and ρ be (not necessarily distinct) primitive elements of $\mathbb{F}_q$, where q is a power of a prime. Then the permutation array $(A_{i,j})$ of order $q - 2$ for which*

$$A_{i,j} = 1 \quad \text{if and only if} \quad \phi^i + \rho^j = 1$$

is a Costas array.

We firstly show that all Welch Costas arrays of order $p - 1 \geq 4$ contain exactly $\frac{(p-1)(p-3)}{8}$ mirror pairs (Theorem 4.16). We then state further results on mirror pairs in Welch Costas and Golomb Costas arrays, whose proofs can be found in [24].

Every Welch Costas array is G-symmetric [13] and, as stated in Theorem 4.10(v), all G-symmetric Costas arrays of even order n contain at least $\frac{n(n-2)}{8}$ mirror pairs. Although this minimum number can be exceeded (see Section 4.1 for examples with $n = 6$ and $n = 8$), we now show that it cannot be exceeded for the subclass of Welch Costas arrays.

Theorem 4.16. *Every Welch Costas array of order $p-1 \geq 4$ contains exactly $\frac{(p-1)(p-3)}{8}$ mirror pairs, namely those specified in Lemma 4.9(i).*

Proof. Let W be a $W_1(p, \phi, c)$ Welch Costas array of order $p - 1 \geq 4$, so that W contains all $\frac{(p-1)(p-3)}{8}$ mirror pairs specified in Lemma 4.9(i). Suppose, for a contradiction, that W also contains some other (w, h)-mirror pair. By Lemma 4.9(ii), both vectors of the mirror pair cross the vertical bisector of W, and by Lemma 4.9(ii) and Theorem 4.10(iv) we may take $w < \frac{p-1}{2}$.

Let the leftmost of the right endpoints of the two vectors of the (w, h)-mirror pair occur in column $j > \frac{p-1}{2}$. The array W' obtained by cyclically shifting the columns of W by $j - \frac{p-1}{2}$ places to the left then contains a (w, h)-mirror pair exactly one of whose vectors crosses its vertical bisector. But W' is the Welch Costas array $W_1(p, \phi, c + j - \frac{p-1}{2})$, which by the same argument as given above cannot contain such a mirror pair. □

The answer to Question 4.4 for Welch Costas arrays is *yes*, by combination of Theorem 4.10(i) with the following corollary to Theorem 4.13.

Corollary 4.17 ([24, Corollary 77]). *For $p \geq 5$, every $W_1(p, \phi, c)$ Welch Costas array contains at least one mirror pair of height 1 unless (p, ϕ, c) belongs to the set*

$$\{(5, 2, 3), (5, 2, 1), (5, 3, 2), (5, 3, 0), (7, 3, 2), (7, 3, 5), (7, 5, 5), (7, 5, 2)\}.$$

Theorem 4.18 gives further restrictions on the mirror pairs of small width and height in a Welch Costas array.

Theorem 4.18 ([24, Theorem 81]). *For $p \geq 5$, every $W_1(p, \phi, c)$ Welch Costas array contains a $(1, 1)$-mirror pair or a $(1, 2)$-mirror pair or a $(2, 1)$-mirror pair.*

Theorem 4.19 gives an analogous result to Theorem 4.18 for the existence of mirror pairs with constrained width and height in a Golomb Costas array In particular, it shows that every Golomb Costas array of order at least 7 contains a $(1, h)$-mirror pair with $h \in \{1, 2, 3\}$. So every Golomb Costas array of order at least 7 contains a mirror pair of width 1, and therefore a mirror pair of height 1 by Remark 4.5 (since the set of Golomb Costas arrays is closed under the transpose operation). This shows that the answer to Question 4.4 for Golomb Costas arrays is *yes*.

Theorem 4.19 ([24, Theorem 83]). *The only Golomb Costas arrays of order $q - 2 \geq 5$ that do not contain a $(1, h)$-mirror pair for all $h \in \{1, 2, 3\}$ are that corresponding to the permutation $[5, 3, 2, 6, 1, 4]$ and its image under horizontal reflection, vertical reflection and $180°$ rotation.*

References

[1] J.K. Beard, *Orders three through 100*. Database of Costas arrays, `http://jameskbeard.com/jameskbeard/Files.html#CostasArrays`.

[2] S.R. Blackburn, T. Etzion, K.M. Martin, and M.B. Paterson, *Two-dimensional patterns with distinct differences — constructions, bounds, and maximal anticodes*, IEEE Transactions on Information Theory **56** (2010), 1216–1229.

[3] C.P. Brown, M. Cenkl, R.A. Games, J. Rushanan, O. Moreno, and P. Pei, *New enumeration results for Costas arrays*, Proceedings of the IEEE International Symposium on Information Theory (Kobe, Japan), 1993.

[4] J.P. Costas, *A study of a class of detection waveforms having nearly ideal range-Doppler ambiguity properties*, Proceedings of the IEEE **72** (1984), 996–1009.

[5] K. Drakakis, *A review of Costas arrays*, Journal of Applied Mathematics (2006), Article ID 26385, 32 pages.

[6] K. Drakakis, *Open problems in Costas arrays*, arXiv 1102.5727v1 [math.CO] (2011).

[7] K. Drakakis, R. Gow, J. Healy, and S. Rickard, *Cross-correlation properties of Costas arrays and their images under horizontal and vertical flips*, Mathematical Problems in Engineering (2008), Article ID 369321, 11 pages.

[8] K. Drakakis, R. Gow, and S. Rickard, *Interlaced Costas arrays do not exist*, Mathematical Problems in Engineering (2008), Article ID 456034, 9 pages.

[9] K. Drakakis, R. Gow, and S. Rickard, *Common distance vectors between Costas arrays*, Advances in Mathematics of Communications **3** (2009), 35–52.

[10] K. Drakakis, F. Iorio, and S. Rickard, *The enumeration of Costas arrays of order 28 and its consequences*, Advances in Mathematics of Communications **5** (2011), 69–86.

[11] P. Erdős, R. Graham, I.Z. Ruzsa, and H. Taylor, *Bounds for arrays of dots with distinct slopes or lengths*, Combinatorica **12** (1992), 39–44.

[12] P. Erdős and P. Turán, *On a problem of Sidon in additive number theory and on some related problems*, Journal of the London Mathematical Society **16** (1941), 212–215.

[13] T. Etzion, *Combinatorial designs derived from Costas arrays*, Sequences: combinatorics, compression, security, and transmission (New York, NY, USA) (R. M. Capocelli, ed.), Springer-Verlag, 1990, pp. 208–227.

[14] T. Etzion, *Sequence folding, lattice tiling, and multidimensional coding*, IEEE Transactions on Information Theory **57** (2011), 4383–4400.

[15] A. Freedman and N. Levanon, *Any two $N \times N$ Costas arrays must have at least one common ambiguity sidelobe if $N > 3$ — a proof*, Proceedings of the IEEE **73** (1985), 1530–1531.

[16] E.N. Gilbert, *Latin squares which contain no repeated digrams*, SIAM Review **7** (1965), 189–198.

[17] S. Golomb, *Algebraic constructions for Costas arrays*, Journal of Combinatorial Theory Series A **37** (1984), 13–21.

[18] S.W. Golomb and H. Taylor, *Constructions and properties of Costas arrays*, Proceedings of the IEEE **72** (1984), 1143–1163.

[19] S. Rickard, *Database of Costas arrays*, `http://osl-vps-4.ucd.ie/downloader`. Accessed 2012.

[20] S. Rickard, *Searching for Costas arrays using periodicity properties*, Proceedings of the 2004 IMA International Conference on Mathematics in Signal Processing (Cirencester, UK).

[21] J.C. Russo, K.G. Erickson, and J.K. Beard, *Costas array search technique that maximizes backtrack and symmetry exploitation*, 44th Annual Conference on Information Sciences and Systems (Princeton, NJ, USA), 2010.

[22] H. Taylor, *Non-attacking rooks with distinct differences*, Technical Report CSI-84-03-02, Communication Sciences Institute, University of Southern California, 1984.

[23] K. Taylor, S. Rickard, and K. Drakakis, *On Costas arrays with various types of symmetry*, 16th International Conference on Digital Signal Processing, IEEE, 2009.

[24] J.L. Wodlinger, *Costas arrays, Golomb rulers and wavelength isolation sequence pairs*, Master's thesis, Simon Fraser University, Department of Mathematics, 2012, `https://theses.lib.sfu.ca/thesis/etd7112`.